ON INTEGRAL EQUATIONS OF THE THEORY OF ELASTICITY WITH DIFFERENCE AND SUMMATION KERNELS

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An approximate method is proposed for solving linear integral equations with kernels dependent on the difference of the arguments (difference kernels), on the sum of the arguments (summation kernels), as well as with kernels representing the superposition of those mentioned (difference-summation kernels). The case of one and two finite intervals, as well as semi-infinite and infinite intervals, is examined. The approximate method proposed is based on reduction of the mentioned integral equations to infinite systems of algebraic equations. In conclusion, a number of problems of the theory of elasticity are mentioned to which the proposed method of solving integral equations can be applied, are indicated.

1. Finite interval. Let us examine the integral equation

$$\varphi(x) + \lambda \int_{-1}^{1} k(x-y)\varphi(y)\,dy = f(x) \qquad (-1 \leqslant x \leqslant 1) \tag{1.1}$$

under the assumptions that

$$k(x) = k(-x), \qquad \int_0^2 |k(x)|^2\,dx = K^2 < \infty \tag{1.2}$$

and that λ is not an eigennumber.

The interval (–1, 1) has been selected to shorten the writing. No generality is hence lost since any finite interval can be reduced to that mentioned by a substitution without altering the properties of the kernel in (1.2).

The substance of the method elucidated as applied to the case of a finite interval is to represent the kernel function $k(x)$ as a series in an orthonormal system of the form

$$C_0^+(x) = 2^{-1/2}, \qquad C_m^+(x) = \cos(m\pi x/2) \quad m = 1, 2, 3, \ldots \tag{1.3}$$

$$k(x) = \sum_{m=0}^{\infty} a_m C_m^+(x) \quad 0 \leqslant x \leqslant 2, \qquad a_m = \int_0^2 k(x) C_m^+(x)\,dx$$

with a subsequent examination of the even $\varphi_+(x)$ and odd $\varphi_-(x)$ solutions of Eq. (1.1).

Representing the right side as the sum of even and odd functions, e. i. $f = f_+ + f_-$, we establish from (1.1)

$$\varphi_\pm(x) + \lambda \int_0^1 K_\pm(x, y)\varphi_\pm(y)\,dy = f_\pm(x) \qquad (0 \leqslant x \leqslant 1) \tag{1.4}$$

$$K_\pm(x, y) = k(x-y) \pm k(x+y) \tag{1.5}$$

Utilizing (1.3), we obtain the following bilinear expansions for these kernels

$$K_\pm(x, y) = 2\sum_{m=0}^{\infty} a_m C_m^\pm(x) C_m^\pm(y), \qquad C_m^-(x) = \sin\frac{m\pi n}{2} \tag{1.6}$$

Substituting (1.6) into (1.4) results in the following formulas for their solutions

$$\varphi_{\pm}(x) = f_{\pm}(x) - 2\lambda \sum_{m=0}^{\infty} a_m C^{\pm}{}_m(x)\, \varphi^{\pm}{}_m \quad , \quad \varphi^{\pm}_m = \int_0^1 C^{\pm}{}_m(x)\, \varphi_{\pm}(x)\, dx \qquad (1.7)$$

The coefficients $\varphi_m^{\pm}$ should hence be found from the infinite systems

$$\varphi_m^{\pm} + \lambda \sum_{m=0}^{\infty} a_m b_{nm}^{\pm} \varphi^{\pm}{}_m = f^{\pm}{}_n \quad (n = 0, 1, 2, \ldots), \qquad f_n^{\pm} = \int_0^1 C^{\pm}{}_m(x)\, f(x)\, dx \qquad (1.8)$$

$$b_{nm}^{\pm} = 2\int_0^1 C^{\pm}{}_m(x)\, C^{\pm}{}_n(x)\, dx = b_{n-m} \pm b_{n+m}, \quad b_k = \frac{2 \sin k\pi/2}{\pi k} \qquad (1.9)$$

Let us turn to an investigation of the infinite systems obtained. Applying the method of proof of complete regularity which is expounded in section 1 of [1], the following complete regularity condition can be established for the infinite systems (1.8):

$$\sqrt{2}\,\lambda K \leqslant 1 - \varepsilon \qquad \left(K^2 = \int_0^2 |k(x)|^2\, dx = \sum_{m=0}^{\infty} |a_m|^2\right) \qquad (1.10)$$

where ε is a fixed small number.

Let us now show that the systems (1.8) are quasi-completely regular for any λ if the following asymptotic expression holds:

$$a_m = O(1/m), \qquad m \to \infty \qquad (1.11)$$

It must be shown that

$$\lim_{n\to\infty} S_n = 0, \qquad S_n = \sum_{m=0}^{\infty} |a_m b_{nm}^{\pm}| \qquad (1.12)$$

In conformity with (1.9) we have

$$S_n \leqslant S_n^{+} + S_n^{-}, \qquad S_n^{\pm} = \sum_{m=0}^{\infty} |a_m|\,|b_{n\pm m}| \qquad (1.13)$$

Let us first estimate S_n^-. To do this we make the substitution $m - n = k$ and represent S_n^- as

$$S_n^{-} = S_n^{(1)} + S_n^{(2)}, \qquad S_n^{(1)} = \sum_{k=}^{\infty} |a_{n+k}|\,|b_k|, \quad S_n^{(2)} = \sum_{m=0}^{\infty} |a_m|\,|b_{n-m}| \qquad (1.14)$$

Let us choose $n > N_0$, where N_0 is so large a fixed number that in conformity with (1.11)

$$|a_m| \leqslant A m^{-1} \quad (m > N_0,\ A > 0)$$

Then

$$S_n^{(2)} \leqslant \sum_{m=0}^{N_0-1} |a_m|\,|b_{n-m}| - \frac{2A}{\pi}\left(\sum_{m=1}^{N_0-1} \frac{1}{m(n-m)} - S_n^{(3)}\right), \qquad S_n^{(3)} = \sum_{m=1}^{n-1} \frac{1}{m(n-m)}$$

Finally, utilizing the identity and the asymptotic relationship [2]

$$\frac{1}{m(n-m)} = \frac{1}{n}\left(\frac{1}{m} + \frac{1}{n-m}\right), \qquad \sum_{k=1}^{n} \frac{1}{k} = O(\ln n), \quad n \to \infty$$

we conclude that $S_n^{(3)} = O(n^{-1} \ln n)$, and therefore

$$S_n^{(2)} = O(n^{-1} \ln n), \quad n \to \infty \qquad (1.15)$$

To estimate the sum $S_n^{(1)}$, we again select $n > N_0$; then

$$S_n^{(1)} \leqslant |a_n| + \frac{2A}{\pi}\sum_{k=1}^{\infty}\frac{1}{k(n+k)}$$

Now, if the known relationships [2] are taken into account

$$\sum_{k=1}^{\infty}\frac{1}{(k+p)(k+q)} = \frac{\psi(q+1)-\psi(p+1)}{q-p}, \qquad \psi(z) = O(\ln z), \quad z\to\infty \qquad (1.16)$$

for the Euler ψ-function, then we perceive that

$$S_n^{(1)} = O(n^{-1}\ln n), \qquad n\to\infty$$

Analogous resonong shows that such an asymptotics holds for S_n^+ also, and therefore, by virtue of (1.15), (1.14) and (1.13) that (1.12) holds, i.e. the systems (1.8) are quasi-completely regular.

Now, in place of (1.11) let there be the more general asymptotic relation

$$a_m = O(m^{-\alpha}), \qquad m\to\infty, \quad \alpha>0 \qquad (1.17)$$

In order to obtain a quasi-complete regular infinite system, the new unknowns $\psi_n^{\pm} = (1+n)^{1-\alpha}\varphi_n^{\pm}$ should be introduced instead of $\varphi_n^{\pm}$; we then obtain from (1.8) the following system:

$$\psi_n^{\pm} + \lambda\sum_{m=0}^{\infty} a^*_m (1+n)^{1-\alpha} b_{nm}^{\pm}\psi^{\pm}_m = (1+n)^{1-\alpha} f^{\pm}_n \qquad (1.18)$$

By virtue of (1.17) the asymptotic expression (1.11) will hold for the coefficients $\overset{*}{a}_m = (1+m)^{\alpha-1}a_m$, therefore, if $(1+n)^{1-\alpha}f_n^{\pm}$ $(n = 0, 1, 2, \ldots)$ are bounded, the infinite system (1.18) is quasi-completely regular. The asymptotic expression for S_n at $\alpha < 1$ is hence worsened and takes the form $S_n = O(n^{-\alpha}\ln n)$. Therefore, the method of truncation can be applied to solve the infinite systems (1.8) or (1.18), which is evidently equivalent to the following approximation of the kernel function:

$$k(x) \approx \sum_{m=0}^{N} a_m C_m^+(x) \qquad (1.19)$$

Evaluation of the coefficients a_m is an important step in the practical utilization of the expounded method of solving (1.1). The formula in (1.3) can turn out to be inconvenient. Moreover, the function $k(x)$ might be given in the form of tabulated values. These complications could be overcome by involving well developed methods of trigonometric interpolation [3]. In this case the coefficients of the approximation (1.19) are expressed by means of a known formula in [3] in terms of discrete values of the function $k(x)$.

The kernel function is often given in the form

$$k(x) = \frac{1}{\pi}\int_0^{\infty} K(t)\cos xt\, dt \qquad (1.20)$$

If, with increasing t, the density $K(t)$ tends to zero more strongly than is determined by the asymptotic expression

$$K(t) = \frac{\Upsilon}{t} + O(t^{-2}), \qquad t\to\infty \qquad (1.21)$$

then the function (1.20) will be continuous and to compute the coefficients a_m in (1.19) the mentioned formulas from the theory of trigonometric interpolation can be used by first calculating the needed values of the function $k(x)$ by utilizing the Filon method

[4], say.

In case the asymptotic (1.21) holds, the singularity in the function $k(x)$ should be isolated, i.e. it should be represented as

$$\pi k(x) = \ln\left|cth\frac{x\pi}{4}\right| + \int_0^\infty \left[K(t) - \frac{tht}{t}\right]\cos xtdt \qquad (1.22)$$

Here, the known integral [2] is utilized

$$\int_0^\infty \frac{tht}{t}\cos xtdt = \ln\left|cth\frac{x\pi}{4}\right| \qquad (1.23)$$

Let us approximate the continuous part of the function (1.22) in the form (1.19) by utilizing trigonometric interpolation formulas [3] to compute the coefficients a_m.

We will have the expansion

$$a_m = \frac{\text{th}\,0.5m\pi}{m} - (-1)^m\frac{4}{\pi}\sum_{k=0}^\infty \frac{e^{-\pi(2k+1)}}{m^2 + (2k+1)^2} \qquad (1.24)$$

for the Fourier coefficients a_m of the function (1.23) as a result of using formulas from (1.3).

In case $m = 0$ the result should be multiplied by $2^{-1/2}$. The series entering here can be expressed in terms of known special functions. However, because of its quite rapid convergence, there is no need for this. A representation of the function (1.23) in the form of a series obtained from its integral representation by replacing $\text{th}\,t$ by its expansion in simple fractions [2] was utilized to obtain (1.24).

2. On an error estimate. The approximateness of the expounded method of solving the integral equation (1.1) occurs because of the approximation (1.19), which is equivalent, according to (1.5), to the following:

$$K_\pm(x, y) \approx 2\sum_{m=0}^N a_m C_m^\pm(x) C_m^\pm(y) = K^*_\pm(x, y) \qquad (2.1)$$

In other words, the exact integral equations (1.4) are replaced by approximate equations. These can be written in the following operator form

$$\varphi + \lambda K\varphi = f, \qquad \varphi^* + \lambda K^*\varphi^* = f \qquad (2.2)$$

(here and throughout in Sect. 2, we omit the plus and minus signs on the symbols).

Considering them to be given in some Banach space, we obtain the error estimate [5]

$$\|\varphi - \varphi^*\| \leqslant q(1-q)^{-1}\|\varphi^*\| \qquad (\lambda\|R_\lambda^*\|\delta \leqslant q < 1), \qquad \delta = \|K - K^*\| \qquad (2.3)$$

The operator R_λ^* here inverts the approximate integral equation, i.e. $\varphi^* = R_\lambda^* f$; it is given by the formula

$$\varphi^*(x) = f(x) - 2\lambda\sum_{m=0}^N a_m\varphi_m C_m(x) \qquad (2.4)$$

where the coefficients φ_m are found from the following system of equations:

$$\varphi_n + \lambda\sum_{m=0}^N a_m b_{nm}\varphi_m = f_n \qquad (2.5)$$

obtained by truncation of the system (1.8).

Let us estimate the norm of the operators in (2.3). Let (2.2) be given in L_2. Then

$$\|K-K^*\|\leqslant 4\int_0^2\int_0^2\Big(\sum_{m=N+1}^{\infty}a_mC_m(x)C_m(y)\Big)^2dxdy\leqslant 4\sum_{m=N+1}^{\infty}a_m^2 \tag{2.6}$$

Therefore, the formula

$$\delta^2/4=\sum_{m=N+1}^{\infty}a_m^2=K^2-\sum_{m=0}^{N}a_m^2 \tag{2.7}$$

can be used to evaluate δ.

The interval of integration was extended to obtain the estimate (2.6), and this could strengthen the inequality to two in order that the orthonormality of $C_m(x)$ could be used. Proceeding analogously further, and drawing upon the Cauchy-Buniakowski inequality where necessary, we obtain

$$\|R_\lambda^*f\|=\|\varphi^*\|\leqslant\|f\|+2|\lambda|\Big(\sum_{m=0}^{N}a_m^2\varphi_m^2\Big)^{1/2}$$

Furthermore, calculating φ_m by Cramer's rule, and estimating the determinant in the numerator according to Hadamard while taking account of the obvious estimates

$$\sum_{k=0}^{N}f_k^2\leqslant\sum_{k=0}^{\infty}f_k^2=\|f\|^2,\qquad \sum_{k=0}^{N}b_{jk}^2\leqslant\sum_{k=0}^{\infty}b_{jk}^2=\int_0^1\Big|2\cos\frac{j\pi x}{2}\Big|dx=2 \tag{2.8}$$

we obtain

$$\|R_\lambda^*\|\leqslant 1+\frac{2|\lambda|}{|\Delta|}\Big(\sum_{k=0}^{N}\frac{a_k^2}{A_k}\prod_{k=0}^{N}A_k\Big)^{1/2}\qquad (A_k=1+2\lambda a_k+2\lambda^2a_k^2) \tag{2.9}$$

Here Δ is the determinant of the system (2.5).

If the kernel of (1.1) is a continuous function, the integral equation (2.2) can be considered in the space of continuous functions. In this case we can estimate the quantity

$$\max\Big|k(x)-\sum_{m=0}^{N}a_mC_m(x)\Big|\leqslant\delta_0\qquad(0\leqslant x\leqslant 2)$$

and thereby the norm of the operator

$$\|K-K^*\|=\max_{0\leqslant x,\,y\leqslant 1}\Big|k(x+y)-\sum_{m=0}^{N}a_mC_m(x+y)+k(x-y)-$$
$$-\sum_{m=0}^{N}a_mC_m(x-y)\Big|\leqslant 2\delta_0$$

Thus, in the case under consideration $2\delta_0$ can be taken as δ in (2.3). In this case we have the estimate

$$\|R_\lambda^*\|\leqslant 1+\frac{2|\lambda|}{|\Delta|}\sum_{k=0}^{N}\frac{|a_k|}{A_k^{1/2}}\Big(\prod_{k=0}^{N}A_k\Big)^{1/2} \tag{2.10}$$

obtained by analogous means for the norm of the operator R_λ^* in the place of (2.9). Hence, besides (2.8) it should still be taken into account that

$$\int_0^1|f(x)^2dx\leqslant\int_0^1|\max f(x)|^2dx=\|f\|^2\qquad(0\leqslant x\leqslant 1)$$

Finally, there is often the case when $k(x)$ is not a continuous function but possesses the property

$$\max\Big|\int_0^1K(x,y)\,dy\Big|=A<\infty\qquad(0\leqslant x\leqslant 1) \tag{2.11}$$

Then, as before, Eqs. (2.2) can be considered in the space of continuous functions [5]. In this case

$$\|K - K^*\| \leqslant \max_{0 \leqslant x \leqslant 1} \left| \int_0^1 K(x, y)\, dy - 2 \sum_{m=0}^{N} a_m C_m(x) \int_0^1 C_m(y)\, dy \right| = \delta \qquad (2.12)$$

and the estimate (2.10) is valid for the norm of the operator $\| R_\lambda^* \|$.

It was assumed above that λ is not an eigennumber of the first equation of (2.2) and the determinant Δ of the system (2.5) is nonzero. In principle, this latter can always be achieved by diminishing the norm of the operator $\|K - K^*\|$ because of increasing N. In this connection it is useful to note that when $\lambda > 0$ and $a_m \geqslant 0$ $(m = 0, 1, 2 \ldots)$, $\Delta \neq 0$ for any N Indeed, if $\Delta = 0$, then this is equivalent to the equation

$$\varphi^*(x) + \lambda \int_0^1 K^*(x, y)\, \varphi^*(y)\, dy = 0 \qquad (2.13)$$

having no trivial solution $\varphi^*(x) \neq 0$. But this is impossible since we obtain, multiplying (2.13) scalarly by $\varphi^*(x)$

$$\| \varphi^* \|^2 + \lambda (\varphi^*, K^*\varphi^*) = 0$$

where $(\varphi^*, K^*\varphi^*) > 0$ because of (2.1) and $a_m > 0$. Therefore, $\Delta \neq 0$ for any N including $N = \infty$. This latter means that for $a_m \geqslant 0$ the integral equations (1.4) can have only negative eigenvalues.

3. Finite interval. Some extensions. Everything expounded above goes over entirely into integral equations of the form

$$\varphi(x) + \lambda \int_{-1}^{1} [k_1(x - y) + k_2(x + y)]\, \varphi(y)\, dy = f(x) \qquad (k_j(x) = k_j(-x),\ j = 1, 2) \qquad (3.1)$$

The method elucidated for the approximate solution of (1.1) is also applicable to equations of the form

$$\varphi(x) + \lambda \rho(x) \int_{-1}^{1} [k_1(x - y) + k_2(x + y)]\, \sigma(y)\, \varphi(y)\, dy = f(x) \qquad (3.2)$$

$$(k_j(x) = k_j(-x);\ j = 1, 2;\ \rho(x) = \rho(-x),\ \sigma(x) = \sigma(-x))$$

However, (1.9) for computation of the coefficients of the appropriate infinite system is hence complicated.

As will be illustrated below, in solving certain contact problems situations will be encountered when the expansion (1.3) of the function $k(x)$ is known in the interval $(-2, 2)$ while Eq. (1.1) is given in the two intervals $(-\alpha, -\beta)$ and (β, α)

$$\varphi(x) + \lambda \left(\int_{-\alpha}^{-\beta} + \int_{\beta}^{\alpha} \right) k(x - y)\, \varphi(y)\, dy = f(x) \qquad (0 \leqslant \beta < \alpha \leqslant 1) \qquad (3.3)$$

In this case we shall have the following equations for the even and odd versions of the problem

$$\varphi_\pm(x) + \lambda \int_\beta^\alpha K_\pm(x, y)\, \varphi_\pm(y)\, dy = f_\pm(x) \qquad (3.4)$$

where the kernels are defined by the previous formulas (1.5) and (1.6), and the solutions by (1.7) and the infinite systems (1.8) or (1.18), but hence

$$f_n^{\pm} = \int_{\beta}^{\alpha} C_n^{\pm}(x) f_{\pm}(x)\,dx, \qquad b_n = \frac{4}{\pi n}\sin\frac{n\pi(\alpha-\beta)}{4}\cos\frac{n\pi(\alpha+\beta)}{4}$$
$$(n = 0, 1, 2, \ldots) \tag{3.5}$$

As above, it can be shown that the corresponding infinite systems will be completely regular if $\lambda(\alpha+\beta)^{1/2}K \leqslant 1-\varepsilon$. Their quasi-complete regularity for any λ is proved analogously.

Everything above relative to (3.3) goes over entirely to an analogous equation with a sum-difference kernel.

It was assumed everywhere above that the kernel function is even. This constraint ca· be avoided if the kernel function is expanded in series of the form

$$k(x) = \frac{1}{2}\sum_{m=-\infty}^{\infty} a_m e^{im\pi x/2}, \qquad a_m = \frac{1}{2}\int_{-2}^{2} k(x)\, e^{im\pi x/2} \tag{3.6}$$

However, the computational algorithm becomes somewhat complicated because of the need to operate with complex numbers.

A more substantial constraint is the requirement that the integral equation be an equation of the second kind. In pure form the elucidated method does not evidently go over to equations of the first kind. It is true that a certain modification has already been utilized in [6] to solve an integral equation of the first kind with a difference kernel. However, it apparently turned out to be only slightly effective. It seems to us that it is expedient to utilize a method based on isolating the singularities in the kernel, particularly the method of orthogonal polynomials [1], in the case of integral equations of the first kind.

4. Semi-infinite and infinite intervals. Let us examine the equation

$$\varphi(x) + \lambda\int_0^{\infty} k(x-y)\,\varphi(y)\,dy = f(x) \qquad (0 \leqslant x < \infty) \tag{4.1}$$

As is known [7], the Fourier transformation $K(t)$ of the kernel function $k(x)$ plays an important part in the solution of the integral equations (4.1), where

$$k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(t)\, e^{-ixt}dt \tag{4.2}$$

Let us reduce (4.1) to an infinite system. To this end, let us construct the solution in the form of a series in Laguerre polynomials

$$\varphi(x) = 2e^{-x}\sum_{m=0}^{\infty} \varphi_m L_m(2x) \tag{4.3}$$

Let us substitute (4.3) into (4.1), after which we integrate both sides of the equation in the interval $(0, \infty)$ with weight $e^{-x}L_n(2x)$. Hence, we will have in place of (4.1)

$$\varphi_n + \lambda\sum_{m=0}^{\infty} a_{n-m}\varphi_m = f_n \quad \left(f_n = \int_0^{\infty} e^{-x} f(x)\, L_n(2x)\,dx\right) \tag{4.4}$$

It turns out that

$$\int_0^{\infty}\int_0^{\infty} \frac{k(x-y)}{e^{x+y}} L_n(2x)\, L_m(2y)\,dxdy = \frac{a_{n-m}}{2}, \quad a_k = \frac{(-1)^k}{\pi}\int_{-\infty}^{\infty} \frac{K(t)}{1+t^2}\left(\frac{1-it}{1+it}\right)^k dt \tag{4.5}$$

This last formula is obtained as a result of utilizing the representation (4.2) and formula 7.414(6) from [2].

Therefore, the integral equation (4.1) will result in its discrete analog (4.4) [7], for which it will sometimes be simpler to obtain an exact solution than for the initial equation. However, basic here is the fact that the infinite system (4.4) is more convenient for an approximate solution. And even more so since (4.5) can be reduced to a form more convenient for calculations. To this end, let us first consider the case when $k(x) = k(-x)$. Then we have in place of (4.2)

$$K(t) = 2\int_0^\infty k(x)\cos xt\,dx, \qquad k(x) = \frac{1}{\pi}\int_0^\infty K(t)\cos xt\,dt \tag{4.6}$$

i.e. $K(t)$ is a real even function, and this permits reduction of the integration in (4.5) to a semi-infinite interval

$$a_k = \frac{(-1)^k}{\pi}\int_0^\infty \frac{K(t)\left[(1+it)^{2k} + (1-it)^{2k}\right]dt}{(1+t^2)^{k+1}} = a_{-k} \tag{4.7}$$

The substitution $t = (\tau^2 - 1)^{1/2}$ and utilization of formula 8.440 from [2] for the Chebyshev polynomial of the first kind $T_n(z)$ lead to the following result:

$$\frac{\pi a_k}{(-1)^k} = \int_{-1}^{1} K\left(\sqrt{\tau^{-2}-1}\right)\frac{T_{2k}(\tau)\,d\tau}{\sqrt{1-\tau^2}} = \int_0^\pi K(\operatorname{tg}\theta)\cos 2k\theta\,d\theta \tag{4.8}$$

Now, let the kernel be skew symmetric, i.e. $k(x) = k(-x)$. Then

$$K(t) = iK^*(t), \qquad K^*(t) = 2\int_0^\infty k(x)\sin xt\,dx, \qquad k(x) = \frac{1}{\pi}\int_0^\infty K^*(t)\sin xt\,dt \tag{4.9}$$

i.e. $K^*(t)$ is a real odd function and we have instead of (4.8)

$$\frac{\pi}{2}\frac{a^k}{(-1)^k} = \int_0^1 K^*\left(\sqrt{\tau^{-2}-1}\right)U_{2k-1}(\tau)\,d\tau = \int_0^{\pi/2} K^*(\operatorname{tg}\theta)\sin 2k\theta\,d\theta$$
$$(a_0 = 0,\ a_k = -a_{-k}) \tag{4.10}$$

Here $U_n(z)$ is a Chebyshev polynomial of the second kind.

In the general case, the function $k(x)$ in (4.1) must be separated into even and odd components, and to use (4.8) for the even and (4.10) for the odd components in the computation of the coefficients of the infinite system (4.4).

By such means the integral equation

$$\varphi(x) + \lambda\int_0^\infty \left[k(x-y) + k_1(x+y)\right]\varphi(y)\,dy = f(x) \tag{4.11}$$

is reduced to the infinite system

$$\varphi_n + \lambda\sum_{m=0}^\infty \left(a_{n-m} + a_{n+m}^{(1)}\right)\varphi_m = f_n \tag{4.12}$$

As before, (4.5) or (4.8), (4.10) are hence valid for calculation of the coefficients a_k corresponding to the difference kernel $k(x-y)$, and for $a_k^{(1)}$ corresponding to the sum kernel $k_1(x+y)$ the following formulas can be obtained analogously:

$$a_k^{(1)} = \frac{(-1)^k}{\pi} \int_{-\infty}^{\infty} \frac{K_1(t)(1-it)^k}{(1+it)^{k+2}} dt, \qquad k = 0, 1, 2, \ldots$$

$$\frac{\pi a_k^{(1)}}{(-1)^k} = \int_{-1}^{1} K_1\left(\sqrt{\tau^{-2}-1}\right) \frac{T_{2k+2}(\tau)\, d\tau}{\sqrt{1-\tau^2}} = \int_0^{\pi} K_1(\operatorname{tg}\theta)\cos(2k+2)\theta d\theta \qquad (4.13)$$

$$\frac{\pi}{2}\frac{a_k^{(1)}}{(-1)^k} = \int_0^1 K_1^*\left(\sqrt{\tau^{-2}-1}\right)U_{2k+1}(\tau)\, d\tau = \int_0^{\pi/2} K_1^*(\operatorname{tg}\theta)\sin(2k+2)\theta d\theta$$

The second formula in (4.13) is valid for the even kernel function $k_1(x)$, where $K_1(t)$ is its cosine transform, and the third for the odd function $k_1(x)$, where $K_1^*(t)$ is its sine transform.

An equation given in an infinite interval (4.14)

$$\varphi(x) + \lambda \int_{-\infty}^{\infty} [k(x-y) + k_1(x+y)]\varphi(y)\, dy = f(x) \qquad \begin{pmatrix} k(x) = k(-x) \\ k_1(x) = k_1(-x) \end{pmatrix}$$

reduces to the particular case of integrating (4.11) if its right side and its solution are separated into even and odd parts just as was done for (1.1).

In elasticity theory problems the right-hand sides of the integral equations are sometimes given in the form of their cosine or sine transforms, i.e.

$$f(x) = \frac{1}{\pi}\int_0^{\infty} F^+(t)\cos xt dt, \qquad f(x) = \frac{1}{\pi}\int_0^{\infty} F^-(t)\sin xt dt \qquad (4.15)$$

In that case, the trigonometric functions in (4.15) should be replaced by corresponding combinations of exponential functions to evaluate the coefficients f_k defined by the formula from (4.4). Then, by using the same methods as in the evaluation of the coefficients a_k, we obtain (4.16)

$$\frac{2\pi f_k}{(-1)^k} = \int_{-1}^{1} F^+\left(\sqrt{\tau^{-2}-1}\right)\frac{T_{2n+1}(\tau) d\tau}{\tau\sqrt{1-\tau^2}} = \int_0^{\pi} F^+(\operatorname{tg}\theta)\frac{\cos(2n+1)\theta}{\cos\theta} d\theta$$

$$\frac{\pi f_k}{(-1)^k} = \int_0^1 F^-\left(\sqrt{\tau^{-2}-1}\right)\frac{U_{2k}(\tau)\, d\tau}{\tau} = \int_0^{\pi/2} F^-(\operatorname{tg}\theta)\frac{\sin(2n+1)\theta}{\cos\theta} d\theta$$

The first formula here corresponds to the first integral representation (3.15), and the second, to the second representation.

Now let us elucidate the regularity condition for the obtained infinite systems (4.4) and (4.12). We shall hence consider the conditions

$$\sum_{k=-\infty}^{\infty} |a_k| = A < \infty, \qquad \sum_{k=0}^{\infty} |a_k^{(1)}| = A_1 < \infty \qquad (4.17)$$

satisfied.

By substituting $m - n = k$ and $m + n = l$ it is then not difficult to estimate the corresponding sums, and to obtain the following complete regularity conditions:

$$\lambda A \leqslant 1 - \varepsilon, \quad \lambda(A + A_1) \leqslant 1 - \varepsilon \qquad (4.18)$$

for the infinite systems (4.4) and (4.12), respectively.

Moreover, it is shown just as easily that the infinite system corresponding to the integral equation (4.11) for $k(x) \equiv 0$ (sum kernel) is quasi-completely regular for any λ.

The method of truncation can be used for an approximate solution of the obtained infinite systems (4.4) and (4.12), and also for Eqs. (4.1), (4.11). The sufficient conditions (4.18) mentioned here for its application can turn out to be burdensome. In this case, it is necessary to turn to [8], where necessary and sufficient conditions are mentioned for the applicability of the method of truncation (reduction) to systems of the form (4.4).

Calculation of the coefficients of the infinite systems evidently plays an essential part in the practical utilization of the approximate method proposed here for the solution of (4.1) and (4.11). The quadratures in (4.4), (4.5), say, can not always be expressed in terms of sufficiently simple functions. In such cases, either the first equalities in (4.8), (4.10), (4.13), (4.16) should be used, the polynomials there should be replaced by their power representations, and the corresponding moments should be computed numerically, or the second equalities in the same formulas should be utilized with the application of the trigonometric interpolation methods [3], as it was done in the preceding sections.

5. Generalization. On equations of the first kind. The method expounded in Sect. 4 is easily carried over to systems of integral equations of the type (4.1) or (4.11). The formulas mentioned there for the computation of the coefficients of the infinite systems remain valid even in this case.

Now, let us consider equations of more general type, which we write, in application of the sum kernel

$$\varphi(x) + \lambda \int_0^\infty \rho(x) k(x+y) \sigma(y) \overline{\varphi(y)}\, dy = f(x) \tag{5.1}$$

The functions here may be complex-valued. If the solution is constructed by means of (4.4), but with complex coefficients φ_n, we arrive at the following infinite system :

$$\varphi_n + \lambda \sum_{m=0}^{\infty} d_{nm} \overline{\varphi_m} = f_n, \qquad \frac{d_{nm}}{2} = \int_0^\infty \int_0^\infty \frac{k(x+y)}{e^{x+y}} \frac{\rho(x)}{\sigma^{-1}(y)} L_n(2x) L_m(2y)\, dx dy \tag{5.2}$$

The previous formula contained in (4.4) is valid for the coefficients f_n. As will be shown below, in specific cases the formula for d_{nm} can be simplified substantially.

Everything that has been elucidated in Sect. 4 can be carried over formally to equations of the first kind. The corresponding infinite systems will hence not contain the isolated coefficient φ_n explicitly, and there will be no λ. The previous formulas are valid for the computation of the coefficients of the infinite systems. However, a disadvantage of the solution in the form (4.3) is that the singularity at $x = 0$ is not extracted. In many problems of elasticity theory and mathematical physics the solutions of the equations

$$\int_0^\infty k(x-y) \varphi(y)\, dy = f(x) \tag{5.3}$$

are unbounded for $x = 0$.

A method is indicated below of how to reduce (5.3) to the infinite system (4.4) and at the same time to extract the singularity in the solution at $x = 0$. Let us consider the kernel function to be even, i.e. the representation (4.7) to hold and

$$K(t) = \frac{\gamma}{t^{1-2\mu}} [1 + O(t^{-1})], \quad t \to \infty \quad (-{}^1/_2 < \mu < {}^1/_2) \tag{5.4}$$

Utilizing the integral for the Macdonald function [2]

$$\frac{K_\mu(|x|)}{|x|^\mu} = \frac{\Gamma(1/2-\mu)}{2^\mu \sqrt{\pi}} \int_0^\infty \frac{\cos xt\,dt}{(1+t^2)^{1/2-\mu}} \tag{5.5}$$

let us represent the kernel of Eq. (5.3) as

$$k(x) = \frac{2^\mu \gamma K_\mu(|x|)}{\sqrt{\pi}\,\Gamma(1/2-\mu)\,|x|^\lambda} + \frac{1}{\pi}\int_0^\infty \left[K(t) - \frac{\gamma}{(1+t^2)^{1/2-\mu}}\right]\cos xt\,dt \tag{5.6}$$

By virtue of the asymptotic expression (5.4), the integral component is a continuous function. Now, if the spectral relationship [9, 1]

$$\frac{\sqrt{2}}{\sqrt{\pi}}\frac{1}{\Gamma(1/2-\mu)}\int_0^\infty \frac{K_\mu(|x-y|)\,e^{-y}}{|x-y|^\mu y^{1/2-\mu}} L_m^{\mu-1/2}(2y)\,dy = \sigma_m e^{-x} L_m^{\mu-1/2}(2x) \tag{5.7}$$

is taken into account, then in conformity with the method of orthogonal polynomials [1], the solution of (5.3) should be constructed in the form of the series

$$\varphi(x) = 2\frac{e^{-x}}{x^{1/2-\mu}}\sum_{m=0}^\infty \frac{\varphi_m}{\sigma_m} L_m^{\mu-1/2}(2x) \qquad \left(\sigma_m = \frac{\Gamma(m+\mu+1/2)}{m!}\right) \tag{5.8}$$

For the coefficients φ_m we obtain the infinite system (4.4) in which we should set

$$\lambda = \frac{1}{\gamma}, \qquad f_n = \frac{1}{\gamma\sigma_n}\int_0^\infty \frac{e^{-x}}{x^{1/2-\mu}} f(x)\, L_n^{\mu-1/2}(2x)\,dx \tag{5.9}$$

$$a_k = \frac{(-1)^k}{\pi}\int_{-\infty}^\infty \left[K(t) - \frac{1}{(1+t^2)^{1/2-\mu}}\right]\left(\frac{1-it}{1+it}\right)^k \frac{dt}{(1+t^2)^{\mu+1/2}}, \qquad a_k = a_{-k}$$

$$\frac{\pi}{2}\frac{a_k}{(-1)^k} = \int_0^1 K(\sqrt{\tau^2-1})\,\frac{T_{2k}(\tau)\,d\tau}{\tau^{1-2\mu}\sqrt{1-\tau^2}} = \int_0^{\pi/2} \frac{K(\operatorname{tg}\theta)}{(\cos\theta)^{1-2\mu}}\cos 2k\theta\,d\theta$$

$(k = 1, 2, 3, \ldots$

$$\frac{\pi a_0}{2} = \int_0^1 K(\sqrt{\tau^2-1})\,\frac{\tau^{2\mu-1}d\tau}{\sqrt{1-\tau^2}} - \gamma\pi = \int_0^{\pi/2}\frac{K(\operatorname{tg}\theta)}{(\cos\theta)^{1-2\mu}}\,d\theta - \gamma\pi$$

Just as above, the formula for f_n can be reduced to a form analogous to (4.16).

6. Illustrations. Shtaerman [10] showed that the plane contact problem of elasticity theory, taking account of the surface structure of the bodies in contact, can be reduced to the integral equation

$$\varphi(x) + \frac{c}{\pi}\int_{-1}^1 \ln\frac{1}{|x-y|}\,\varphi(y)\,dy = f(x), \qquad \varphi(x) = p(ax) \tag{6.1}$$

Here $p(x)$ is the contact stress under the stamp, $2a$ the length of the contact portion, c the Shtaerman constant, $f(x)$ a function determining the stamp configuration to the accuracy of a constant. A mixed problem of heat conduction reduces to this same equation [11].

In application to (6.1), the coefficients a_m of the infinite system (1.8) have the form

$$a_m = 2(m\pi)^{-1}[\mathrm{Si}(m\pi) + \pi/2] \quad (m=1,2,3,\ldots), \qquad a_0 = \sqrt{2}(1-\ln 2)$$

where $\mathrm{Si}(x)$ is the sine integral. As is seen, condition (1.11) is satisfied in the case under consideration, and therefore, the infinite system (1.8) is quasi-completely regular for any c. It is easy to elucidate that it is completely regular for $c < 1.5$ if (1.10) is taken into account and also

$$K^2 = \int_0^2 \left| \ln \frac{1}{x} \right|^2 dx = 2.188$$

To confirm the efficiency of the proposed method of solving the integral equations, appropriate calculations were carried out in application to the case of impression of a stamp with a flat base, i.e. when $f(x) = B$ in (6.1). The constant B is found from the equilibrium condition of the stamp. It turns out that the contact stresses at $c = 0.1$ differ only in the third digit in all approximations starting with $N = 1$ at all points of the interval, while at $c = 1$ the difference is one in the second digit starting with $N = 3$. Analogous calculations were performed also on the basis of the method proposed in (10], as well as the method mentioned in the last section in [12]. It turns out that they are not as good as that proposed herein in either rapidity of convergence, or in quantity of calculations.

The error was also computed by means of (2.3), (2.7) and (2.9), as well as (2.10) and (2.12). In this case the kernel of the equation satisfies condition (2.11). It turns out that these formulas, in application to (6.1), yield an error with a high accumulation.

The problem of impression of a stamp into the endface of an infinite strip was considered in [13]. If the mentioned problem is solved taking account of the surface structure of the bodies in contact in the Shtaerman formulation [10], we arrive at the equation

$$\varphi(x) + \frac{c}{\alpha\pi} \int_{-\alpha}^{\alpha} \ln \left| \operatorname{ctg} \frac{\pi(x-y)}{4} \right| \varphi(y)\, dy = f(x), \qquad \varphi(x) = p(lx), \qquad \alpha = \frac{a}{l} \qquad (6.2)$$

Here $2l$ is the width of the strip. The remaining symbols are the same as in the preceding problem.

The integral equation (6.2) is a particular case ($\beta = 0$) of Eq. (3.3). According to 1.442 of [2], we have the expansion

$$\ln \operatorname{ctg} \frac{\pi x}{4} = 2 \sum_{m=0}^{\infty} \frac{\cos(m + 1/2)\pi x}{2m+1} \qquad (0 \leqslant x \leqslant 2)$$

and therefore the coefficients a_m in the case under consideration are determined by the simple formulas

$$a_k = 0 \quad (k = 0,1,2,\ldots), \qquad a_{2k+1} = \frac{2}{2k+1} \quad (k=0,1,2,\ldots) \qquad (6.3)$$

Therefore, condition (1.11) is also satisfied here, and therefore, the infinite systems (1.8) corresponding to (6.2), whose elements are defined by (6.3) and (3.5) at $\beta = 0$, are quasi-completely regular for any c. The condition of complete regularity is easily obtained also if it is taken into account that

$$K^2 = 4 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{2}$$

As has been shown in [14], problems of the torsion of an elastic half-space with a spherical inclusion by a stamp are reduced to the following integral equations:

$$\varphi(s) - \frac{2}{\pi} \int_{\alpha_0}^{\pi} \varphi(t) [\eta(t-s) - \eta(t+s)]\, dt = f(s) \qquad (\alpha_0 \leqslant x \leqslant \pi)$$

$$\varphi(s) - \frac{2}{\pi}\int_0^{\alpha_0} \varphi(t)\,[\eta(t-s) - \eta(t+s)]\,dt = f(s) \qquad (0 \leqslant x \leqslant \alpha_0) \tag{6.4}$$

$$\eta(u) = \sum_{n=0}^{\infty} \frac{\cos(n + {}^1/_2)\,u}{\exp[2(n + {}^1/_2)\beta_0] + 1} \qquad (0 \leqslant u \leqslant 2\pi)$$

Here $\varphi(s)$ are the required functions in terms of which the stresses and displacements of the half-space are expressed, $f(s)$ are given functions (to the accuracy of constants) for which there are expressions herein, α_0, β_0 are parameters connected with the stamp radius b, with the radius of the spherical inclusion ρ and with the distance l between the center of this latter and the surface of the half-space, by the dependencies

$$l = \rho \operatorname{ch} \beta_0, \qquad l \operatorname{ctg} {}^1/_2\, \alpha_0 = b \operatorname{cth} \beta_0$$

The first equation from (6.4) corresponds to the case when there are no displacements on the surface of the spherical inclusion (absolutely rigid inclusion), and the second equation corresponds to the case of a spherical cavity on whose surface there are no stresses.

In order to reduce the equations mentioned to the Eqs. (3.4), the substitution $s = \pi x$, $t = \pi y$ must be made. Then the first equation will correspond to the odd case, i.e. the minus sign in (3.4), where $\alpha = 1$, $\beta = \alpha_0 / \pi$ and the second equation to the even case in which $\beta = 0$, $\alpha = \alpha_0 / \pi$. The infinite system (1.8) with minus sign will correspond to the first integral equation of (6.4); $\lambda = -2$, and the coefficients f_n, b_n should be calculated by means of (3.5) setting $\alpha = 1$, $\beta = \alpha_0 / \pi$. In conformity with the expansion from (6.4), the simple formulas

$$a_{2k} = 0, \qquad a_{2k+1} = \{\exp[(2k+1)\beta_0] + 1\}^{-1} \qquad (k = 0, 1, 2, \ldots) \tag{6.5}$$

are valid for the coefficients a_m.

The infinite system (1.8) with plus sign will correspond to the second integral equation of (6.4), where as for the first equation $\lambda = -2$, and (6.5) are valid. As before, the coefficients f_n, b_n should be calculated by means of (3.5), but now $\beta = 0$, $\alpha = \alpha_0 / \pi$.

It has been shown in the monograph [15] that such elasticity theory problems as the impression of a circular stamp into an elastic layer, as well as the torsion of this layer by a circular stamp, the impression of an annular stamp into an elastic half-space, and also a number of mixed problems for the torsion of a truncated sphere are reduced to (1.4).

Let us present several illustrations of the equations in a semi-infinite interval. It has been shown in [16] that the problem of impression of a semi-infinite stamp taking account of the surface structure can be reduced to the equation

$$\varphi(x) + \lambda \int_0^{\infty} K_0(|x-y|)\,\varphi(y)\,dy = f(x) \tag{6.6}$$

Taking account of the integral representation for the Macdonald function (5.5), we find that in this case $K(t) = \pi(1 + t^2)^{-1/2}$. Substituting this expression into (4.5) and utilizing 8.381(1) from [2], we obtain

$$a_k = \frac{2}{1 - 4k^2} = a_{-k} \qquad (k = 0, 1, 2, \ldots) \tag{6.7}$$

Therefore, the integral equation (6.6) reduces to the infinite system (4.4) for whose coefficients the simple formula (6.7) is valid. It is completely regular if $\lambda < 0.25$ since in the case under consideration

$$A = \sum_{k=-\infty}^{\infty} |a_k| = \sum_{k=-\infty}^{\infty} \left| \frac{2}{1-4k^2} \right| = 4$$

A problem from electromagnetic wave theory [17] also reduces to the integral equation (6.6). An exact solution of (6.6) was obtained in this paper, which it is difficult to complete numerically. An approximate method was mentioned in [18] for the solution of the same equation. This method is substantially equivalent to truncating the infinite system obtained here for it.

The problem of impression of a circular stamp in an elastic half-space taking account of cohesion is reduced in [19] to the integral equation

$$\varphi(x) + \frac{\lambda}{\pi} \int_0^\infty \frac{\sin(x-y)}{x-y} \varphi(y)\, dy = f(x) \tag{6.8}$$

In this case evidently

$$k(x) = \frac{\sin x}{x} = \frac{1}{\pi} \int_0^\infty K(t) \cos xt dt, \qquad K(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases}$$

Taking this latter into account, we find on the basis of (4.8)

$$a_k = (-1)^k (2k\pi)^{-1} \sin(\pi k / 2) \qquad (k = 0, 1, 2, \ldots) \tag{6.9}$$

Therefore, the integral equation (6.8) can be reduced to the infinite system (4.4) whose coefficients are determined by the simple formula (6.9).

It has been shown in [20] that the fundamental plane problems of elasticity theory for simply-connected bodies can be reduced to the equation

$$\varphi(x) + \lambda \int_0^\infty k(x+y)\, y\, \overline{\varphi(y)}\, dy = f(x) \tag{6.10}$$

which is a particular case ($\rho = 1,\ \sigma = y$) of (5.1). Thus (6.10) can be reduced to the infinite system (5.2). The formula mentioned there for computation of its coefficients simplifies substantially in the case under consideration. Indeed, if it is taken into account that [2]

$$L_m(y) = L_m^1(y) - L_{m-1}^1(y)$$

as well as that not the kernel itself is given in (6.10) [20], but its Fourier transform $K(t)$, i.e. if (4.2) is taken into account, then we may write

$$d_{nm} = (m+1) c_{n,m} - m c_{n,m-1}, \quad c_{n,m} = a_{n+m} \qquad a_k = \frac{(-1)^k}{2\pi} \int_{-\infty}^{\infty} \frac{K(t)}{(1+it)^3} \left(\frac{1-it}{1+it} \right)^k dt \tag{6.11}$$

Here we should take $c_{n,-1} = 0$. Moreover, it should be taken into account that formula 7.414(8) of [2] was utilized in obtaining the second formula in (6.11). The formula obtained for a_k when the integral it contains is not expressed in terms of simple functions can be subjected to further simplification based on the representations

$$K(t) = K^+(t) + K^-(t), \qquad 2K^\pm(t) = K(t) \pm K(-t)$$

Then, proceeding exactly as in the case of the integral (4.5), we obtain

$$a_k = a_k^+ + a_k$$

$$\frac{\pi}{2} \frac{a_k^+}{(-1)^k} = \int_0^1 K^+ \left(\sqrt{\tau^{-2} - 1} \right) \frac{\tau T_{2k+3}(\tau)\, d\tau}{\sqrt{1-\tau^2}} = \int_0^{\pi/2} K^+(\operatorname{tg} \theta) \cos\theta \cos(2k+3)\theta d\theta$$

$$\frac{\pi}{2}\frac{a_k^-}{(-1)^k}=\int_0^1 K^-(\sqrt{\tau^{-2}-1})\,\tau U_{2k+3}(\tau)\,d\tau=\int_0^{\pi/2} K^-(\operatorname{tg}\theta)\cos\theta\sin(2k+3)\,\theta d\theta$$

The problem of the bending of a semi-infinite plate situated on an elastic half-space was reduced in [9] to the integral equation

$$\int_0^\infty [K_0(|x-y|)+\mu G(x-y)]\,\varphi(y)\,dy=f(x)\qquad \left(G(t)=\frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{-its}ds}{(1+s^2)^2}\right)$$

Utilizing the method described above for reducing an equation of the first kind to infinite systems based on extracting the singularities in the kernel, we arrive at the infinite system (4.4), where the coefficients of this system have the simple form

$$\pi a_k = 12\,(1-4k^2)^{-1}\,(9-4\,k^2)^{-1}$$

Calculations performed in application to this problem showed the high efficiency of the proposed method. These materials, as well as both a more detailed examination of other problems such as those mentioned above, and new problems, will be published later.

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CONTACT PROBLEMS FOR AN ELASTIC SEMI-INFINITE CYLINDER

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A system of homogeneous solutions is constructed for the axisymmetric mixed problem of elasticity theory for an infinite cylinder, one part of whose surface is stress-free, while the other is under sliding boundary conditions. Asymptotic formulas governing the stress concentration and the shape of the free surface at the line of boundary condition separation are obtained. The system can be utilized to satisfy conditions on the endfaces of a semi-infinite or finite cylinder.

Three contact problems of a semi-infinite cylinder partially compressed without friction by an absolutely rigid collar. The conditions on the side surface are hence satisfied exactly. The coefficients in the series of homogeneous solutions are determined from the normal systems of algebraic equations.

1. Let us consider a system of homogeneous solutions each of which satisfies mixed conditions on the surface of cylinder of unit radius

$$\tau_{rz} = u = 0 \quad \text{for} \quad r = 1, \ z \geqslant 0 \tag{1.1}$$

$$\tau_{rz} = \sigma_r = 0 \quad \text{for} \quad r = 1, \ z < 0 \tag{1.2}$$

and has finite elastic stress energy at the line of separation of these conditions at $r = 1$

$$\sigma_r \sim O(z^{\alpha_1 - 1}) \quad \text{for} \quad z \to +0, \qquad u \sim O(z^{\alpha_2}) \quad \text{for} \quad z \to -0 \tag{1.3}$$

$$(\alpha_1, \ \alpha_2 > 0)$$

Let us start with the construction of a subsystem of solutions which increase without